Vectors in Action

SnackLab Practice Set · Vectors · 9 applied problems with solutions · snacklab.dev

Original practice problems written by SnackLab, organized by topic: the dot product, the cross product, and planes, lines and the Gram–Schmidt process. Every answer was checked numerically. Each problem is shown with vector arrows in a SnackLab short; try it yourself in the SnackLab app (v→ mode).

Dot Product in Action

1. Work done by a pull. A box is pulled with force F = (8, a) N while it moves d = (5, 0) m along the floor. Find the work W = F · d and explain why it does not depend on a. For a = 6, find the angle between F and d.

2. Weight on a ramp. A 10 N weight w = (0, −10) rests on a ramp in the direction d = (4, 3). Find the projection of w onto d, its length, and the component of w perpendicular to the ramp.

3. Diagonal of a cube. A box has edges along the axes with lengths 1, 1 and a. Find the angle between the space diagonal v = (1, 1, a) and the edge e = (1, 0, 0) as a function of a, and its value for a cube (a = 1).

Cross Product in Action

4. Torque on a wrench. A force $F = (a, 3, 0)$ N acts at the end of a wrench $r = (2, 0, 0)$ m. Find the torque $\tau = r \times F$ and show that it does not depend on a.

5. Area of a triangle. Find the area of the triangle with vertices $P(1, 0, 0)$, $Q(0, 2, 0)$, $R(0, 0, a)$ using the cross product, and evaluate it for $a = 3$.

6. Volume and coplanarity. Find the volume of the parallelepiped spanned by $u = (1, 2, 1)$, $v = (2, 0, 1)$, $w = (1, a, 0)$ for $a = 2$, and find the value of a for which the three vectors lie in one plane.

Planes, Lines & Gram–Schmidt

7. Point to plane. Find the distance from P(3, 1, a) to the plane $2x - y + 2z = 6$ by projecting QP onto the normal vector, where Q(3, 0, 0) is a point of the plane. Evaluate it for $a = 4$.

8. Two planes meet. Find a vector equation of the line where the planes $x + y + z = 3$ and $x - y + 2z = 2$ meet.

9. Gram–Schmidt. Apply the Gram–Schmidt process to $v_1 = (1, 1, 0)$ and $v_2 = (1, 0, 1)$ to get an orthonormal pair u_1, u_2.

Solutions

Dot Product in Action

1. Work done by a pull

- $W = F \cdot d = 8 \cdot 5 + a \cdot 0 = 40$ J.
- Only the component of F along d does work; the vertical part a is perpendicular to d.
- For $a = 6$: $|F| = 10$, $|d| = 5$, $\cos\theta = 40/50 = 0.8$.
- $\theta = \cos^{-1}(0.8) \approx 36.9°$.

> Answer: $W = F \cdot d = 8 \cdot 5 + a \cdot 0 = 40$ J / Any a gives 40 J: the vertical part does no work

2. Weight on a ramp

- $w \cdot d = 0 \cdot 4 + (-10) \cdot 3 = -30$, $|d|^2 = 25$.
- $\text{proj}_d\, w = (-30/25)(4, 3) = (-4.8, -3.6)$, length 6 N (down the ramp).
- Normal part: $w - \text{proj} = (4.8, -6.4)$, length 8 N.
- Check: $6^2 + 8^2 = 10^2$.

> Answer: $\text{proj}_d\, w = (w \cdot d/|d|^2)\, d = (-30/25)(4, 3) = (-4.8, -3.6)$ / Along the ramp 6 N, pressing into it 8 N

3. Diagonal of a cube

- $e \cdot v = 1$, $|e| = 1$, $|v| = \sqrt{(2 + a^2)}$.
- $\cos\theta = 1/\sqrt{(2 + a^2)}$.
- Cube $a = 1$: $\cos\theta = 1/\sqrt{3} \to \theta \approx 54.7°$.
- As a grows the diagonal turns toward the z axis and $\theta \to 90°$.

> Answer: $\cos\theta = e \cdot v/(|e||v|) = 1/\sqrt{(2 + a^2)}$ / Cube $a = 1$: $\cos\theta = 1/\sqrt{3} \to \theta \approx 54.7°$

Cross Product in Action

4. Torque on a wrench

- $r \times F = (0 \cdot 0 - 0 \cdot 3,\ 0 \cdot a - 2 \cdot 0,\ 2 \cdot 3 - 0 \cdot a) = (0, 0, 6)$.
- The x-part a is parallel to r, and a parallel force gives zero cross product.
- $|\tau| = 6$ N·m, pointing along +z (counter-clockwise seen from above).
- Only the perpendicular part 3 N times the arm 2 m matters.

> Answer: $\tau = r \times F = (0, 0, 2 \cdot 3 - 0 \cdot a) = (0, 0, 6)$ / $|\tau| = 6$ N·m for every a (the part along r gives no torque)

5. Area of a triangle

- PQ = (−1, 2, 0), PR = (−1, 0, a).
- PQ × PR = (2a, a, 2).
- For a = 3: (6, 3, 2), |n| = √(36 + 9 + 4) = 7.
- Area = 7/2; n is also a normal vector of the plane 6x + 3y + 2z = 6.

> Answer: n = PQ × PR = (2a, a, 2) → (6, 3, 2) / Area = |n|/2 = 7/2

6. Volume and coplanarity

- v × w = (0·0 − 1·a, 1·1 − 2·0, 2a − 0) = (−a, 1, 2a).
- u · (v × w) = −a + 2 + 2a = a + 2.
- a = 2: volume |4| = 4.
- Coplanar when the triple product is 0: a = −2.

> Answer: v × w = (−a, 1, 2a) → u · (v × w) = a + 2 / a = 2: volume 4; a = −2: volume 0 (coplanar)

Planes, Lines & Gram–Schmidt

7. Point to plane

- n = (2, −1, 2), |n| = 3; QP = (0, 1, a).
- n · QP = −1 + 2a.
- Distance = |n · QP|/|n| = |2a − 1|/3.
- a = 4: 7/3; the projection of QP on n is (7/9)(2, −1, 2).

> Answer: d = |n · QP|/|n| = |2a − 1|/3 / a = 4: d = 7/3

8. Two planes meet

- The line is perpendicular to both normals n_1 = (1, 1, 1), n_2 = (1, −1, 2).
- d = n_1 × n_2 = (1·2 − 1·(−1), 1·1 − 1·2, 1·(−1) − 1·1) = (3, −1, −2).
- A point: z = 0 gives x + y = 3, x − y = 2 → (5/2, 1/2, 0).
- Line: r(t) = (5/2, 1/2, 0) + t(3, −1, −2).

> Answer: d = n_1 × n_2 = (3, −1, −2) / Line: (x, y, z) = (5/2, 1/2, 0) + t(3, −1, −2)

9. Gram–Schmidt

- u_1 = v_1/|v_1| = (1, 1, 0)/√2.
- proj = (v_2 · v_1/|v_1|²) v_1 = (1/2)(1, 1, 0) = (1/2, 1/2, 0).
- w_2 = v_2 − proj = (1/2, −1/2, 1); check w_2 · v_1 = 0.
- u_2 = w_2/|w_2| = (1, −1, 2)/√6.

> Answer: w_2 = v_2 − (v_2·v_1/|v_1|²)v_1 = (1/2, −1/2, 1) / u_1 = (1, 1, 0)/√2, u_2 = (1, −1, 2)/√6