Series Solutions in Action

SnackLab Practice Set · Series solutions of ODEs · 9 applied problems with solutions · snacklab.dev

Original practice problems written by SnackLab, organized by topic: power series solutions, the Frobenius method, and Bessel and Legendre functions. Every answer was checked symbolically or numerically. Each problem is shown graphically in a SnackLab short; try it yourself in the SnackLab app (d/dx mode).

Power Series Solutions

1. Light in a graded layer. In a layer whose refractive index changes linearly, the wave amplitude obeys the Airy equation $y'' = xy$ with $y(0) = 1$, $y'(0) = 0$. Find the power series solution about $x = 0$, estimate $y(1)$ with three terms, and locate the first zero for $x < 0$.

2. Spreading heat pulse. The profile of a spreading heat pulse satisfies $y'' + xy' + y = 0$ with $y(0) = 1$, $y'(0) = 0$. Find the power series solution, identify it in closed form and compute $y(1)$.

3. Quantum oscillator. The quantum harmonic oscillator leads to the Hermite equation $y'' - 2xy' + 2\lambda y = 0$. With $y(0) = 1$, $y'(0) = 0$, find the recurrence for the series coefficients, show that the series terminates exactly when λ is an even integer, and give the solution for $\lambda = 2$.

Frobenius Method

4. Half-order Bessel. Solve $x^2y'' + xy' + (x^2 - 1/4)y = 0$ by the Frobenius method. Find the indicial roots and the solution with $y(\pi/2) = \sqrt{(2/\pi)}$, $y'(\pi/2) = -(1/2)(2/\pi)^{3/2}$. Where are its zeros?

5. Roots 0 and 1/2. Use the Frobenius method on $4xy'' + 2y' + y = 0$. Find the indicial roots, the two series solutions in closed form, and the solution with $y(1) = \cos 1$, $y'(1) = -(\sin 1)/2$. Where is its first zero?

6. Hydrogen atom. The radial part of the hydrogen atom leads to the Laguerre equation $xy'' + (1 - x)y' + ny = 0$. Show that $x = 0$ is a regular singular point with a double indicial root, find the recurrence for the regular solution with $y(0) = 1$, and give the polynomial for $n = 2$ and its zeros.

Bessel & Legendre Functions

7. Drum head. The radial shape of a vibrating circular drum of radius 1 satisfies $y'' + y'/x + k^2 y = 0$ with $y(0) = 1$, $y'(0) = 0$, and the rim is clamped: $y(1) = 0$. Find the allowed k values for the first two modes and the ratio of their frequencies.

8. Swinging chain. Small swings of a hanging chain of length 1 (x measured up from the free end) satisfy $xy'' + y' + \lambda y = 0$. The top $x = 1$ is fixed ($y(1) = 0$) and the free end is finite ($y(0) = 1$). Solve with the substitution $s = 2\sqrt{(\lambda x)}$ and find the smallest λ.

9. Potential of a sphere. A sphere of radius 1 has surface potential $V = \cos^2\theta$. Inside, $V = \Sigma\, c_n r^n P_n(\cos\theta)$, where P_n solves the Legendre equation $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$. Explain why n must be an integer, expand $\cos^2\theta$ in Legendre polynomials and find V at the center.

Solutions

Power Series Solutions

1. Light in a graded layer

- Put $y = \Sigma\, a_n x^n$: $\Sigma\,(n+2)(n+1)a_{n+2}x^n = \Sigma\, a_{n-1}x^n$, and $a_2 = 0$.
- Recurrence $a_{n+2} = a_{n-1}/((n+2)(n+1))$; with $a_0 = 1$, $a_1 = 0$: $a_3 = 1/6$, $a_6 = 1/180$, $a_9 = 1/12960$.
- $y = 1 + x^3/6 + x^6/180 + x^9/12960 + \ldots$; $y(1) \approx 1 + 0.1667 + 0.0056 = 1.1722$ (exact 1.1723).
- For $x > 0$ the solution grows; for $x < 0$ it oscillates. First zero $x \approx -1.99$.

> Answer: $a_{n+2} = a_{n-1}/((n+2)(n+1))$ → $y = 1 + x^3/6 + x^6/180 + \ldots$ / $y(1) \approx 1.1722$ (3 terms already 1.1722), first zero $x \approx -1.99$

2. Spreading heat pulse

- $y = \Sigma\, a_n x^n$: $(n+2)(n+1)a_{n+2} + n\, a_n + a_n = 0$.
- So $a_{n+2} = -a_n/(n+2)$; $a_2 = -1/2$, $a_4 = 1/8$, $a_6 = -1/48$, odd terms vanish.
- $y = \Sigma\, (-1)^k\, x^{2k}/(2^k k!) = e^{-x^2/2}$.
- $y(1) = e^{-1/2} \approx 0.607$.

> Answer: $a_{n+2} = -a_n/(n+2)$ → $y = 1 - x^2/2 + x^4/8 - x^6/48 + \ldots$ / $y = e^{-x^2/2}$, $y(1) = e^{-1/2} \approx 0.607$

3. Quantum oscillator

- $y = \Sigma\, a_n x^n$: $(n+2)(n+1)a_{n+2} - 2n\, a_n + 2\lambda\, a_n = 0$.
- $a_{n+2} = 2(n - \lambda)a_n/((n+2)(n+1))$.
- With only even terms, the series stops when $\lambda = n$ for some even n; otherwise it grows like e^{x^2}.
- $\lambda = 2$: $a_2 = -2$, $a_4 = 0 \rightarrow y = 1 - 2x^2$ $(= -H_2/2)$.

> Answer: $a_{n+2} = 2(n - \lambda)a_n/((n+2)(n+1))$ → stops when $\lambda = n$ / $\lambda = 2$ → $y = 1 - 2x^2$ (Hermite H_2 up to a constant)

Frobenius Method

4. Half-order Bessel

- $y = \Sigma\, a_n x^{n+r}$: indicial equation $r^2 - 1/4 = 0 \to r = \pm 1/2$.
- $r = 1/2$ gives $x^{1/2}(1 - x^2/3! + x^4/5! - \ldots) = \sin x/\sqrt{x}$; $r = -1/2$ gives $\cos x/\sqrt{x}$.
- The initial values match $y = \sin x/\sqrt{x}$ (≈ 0.798 and -0.254 at $\pi/2$).
- Zeros where $\sin x = 0$: $x = \pi, 2\pi, 3\pi, \ldots$ (amplitude decays like $1/\sqrt{x}$).

> Answer: $r^2 - 1/4 = 0 \to r = \pm 1/2$ / $y = \sin x/\sqrt{x}$, zeros $x = \pi, 2\pi, 3\pi$

5. Roots 0 and 1/2

- Indicial equation $4r(r-1) + 2r = 2r(2r-1) = 0 \to r = 0, 1/2$.
- $r = 0$: $1 - x/2! + x^2/4! - \ldots = \cos\sqrt{x}$; $r = 1/2$: $\sqrt{x} - x^{3/2}/3! + \ldots = \sin\sqrt{x}$.
- The initial values pick $y = \cos\sqrt{x}$.
- First zero $\sqrt{x} = \pi/2 \to x = \pi^2/4 \approx 2.47$.

> Answer: $4r(r-1) + 2r = 0 \to r = 0, 1/2$ / $y = \cos\sqrt{x}$ (other: $\sin\sqrt{x}$), first zero $x = \pi^2/4 \approx 2.47$

6. Hydrogen atom

- Indicial equation $r^2 = 0 \to r = 0$ (double); the second solution contains $\ln x$.
- $y = \Sigma\, a_k\, x\hat{}k$: $(k+1)^2 a_{k+1} = (k-n)a_k$.
- For integer n the series stops at $x\hat{}n$ (Laguerre polynomial).
- $n = 2$: $a_1 = -2$, $a_2 = 1/2 \to y = 1 - 2x + x^2/2$, zeros $x = 2 \pm \sqrt{2} \approx 0.59, 3.41$.

> Answer: $a_{k+1} = (k-n)a_k/(k+1)^2 \to$ stops when n is an integer / $n = 2$: $y = 1 - 2x + x^2/2$, zeros $2 \pm \sqrt{2}$

Bessel & Legendre Functions

7. Drum head

- Bessel equation of order 0 in kx: $y = J_0(kx)$ (Y_0 blows up at the center).
- Clamped rim: $J_0(k) = 0 \to k_1 = 2.405$, $k_2 = 5.520$, $k_3 = 8.654$.
- Frequency $\propto k$, so $\omega_2/\omega_1 = 5.520/2.405 \approx 2.30$.
- Not an integer ratio → a drum sounds less "musical" than a string.

> Answer: $y = J_0(kx)$; $y(1) = 0 \to k = 2.405$ (first zero of J_0) / Next: $k = 5.520 \to$ frequency ratio ≈ 2.30 (not harmonic)

8. Swinging chain

- With $s = 2\sqrt{(\lambda x)}$ the equation becomes Bessel of order 0: $y = J_0(2\sqrt{(\lambda x)})$.
- Finite at $x = 0$ rules out Y_0; also $y'(0) = -\lambda$.
- $y(1) = 0 \to 2\sqrt{\lambda} = 2.405 \to \lambda = (2.405/2)^2 \approx 1.446$.
- Frequency $\omega = \sqrt{(\lambda g/L)}$ for a chain of length L.

> Answer: $y = J_0(2\sqrt{(\lambda x)})$ / $y(1) = 0 \to 2\sqrt{\lambda} = 2.405 \to \lambda \approx 1.446$

9. Potential of a sphere

- Series solution: $a_{k+2} = (k - n)(k + n + 1)a_k/((k + 2)(k + 1))$; unless n is an integer it diverges at $x = \pm 1$.
- Integer n: the series stops → Legendre polynomial; $P_2 = (3x^2 - 1)/2$.
- $\cos^2\theta = 1/3 + (2/3)P_2(\cos\theta)$ → $V = 1/3 + (2/3)r^2P_2(\cos\theta)$.
- At the center $r = 0$: $V = 1/3$ (the average over the sphere).

> Answer: Series stops only for integer n → $P_n(x)$ / $\cos^2\theta = (1/3)P_0 + (2/3)P_2$ → $V(r, \theta) = 1/3 + (2/3)r^2P_2$, center 1/3