Higher-Order ODEs in Action

SnackLab Practice Set · Higher-order differential equations · 18 applied problems with solutions · snacklab.dev

Original practice problems written by SnackLab, organized by topic: constant-coefficient equations, nonhomogeneous equations, springs and circuits, nonlinear equations, boundary-value problems and linear systems. Every answer was checked symbolically or numerically. Each problem is shown graphically in a SnackLab short; try it yourself in the SnackLab app (d/dx mode).

Constant-Coefficient Equations

1. Door closer. A hydraulic door closer obeys $\theta'' + 5\theta' + 6\theta = 0$, where $\theta(t)$ is the opening angle (rad). The door is released from rest at $\theta(0) = 1.2$. Find $\theta(t)$ and decide whether the door ever swings past the closed position $\theta = 0$.

2. Building sway. The top of a tall building, pushed by a sudden gust, moves according to $x'' + 2x' + x = 0$ with $x(0) = 0$ and $x'(0) = 3$. Find $x(t)$, the largest displacement and the time it occurs.

3. Potential between spheres. Between two concentric spheres the electric potential satisfies the Cauchy–Euler equation $r^2V'' + 2rV' = 0$. The inner sphere ($r = 1$) is held at 4 V and the outer sphere ($r = 2$) is grounded. Find $V(r)$.

Nonhomogeneous Equations

4. Bead sinking in oil. A small bead released from rest sinks in oil with drag proportional to its speed: $y'' = 2 - 2y'$, $y(0) = 0$, $y'(0) = 0$ (scaled units, y downward). Find $y(t)$, the depth at $t = 4$ and the terminal speed.

5. Marching on a bridge. A footbridge deck with natural frequency 3 is driven by marching at frequency 2: $x'' + 9x = 4\cos 2t$, $x(0) = 0$, $x'(0) = 0$. Find $x(t)$ and its largest displacement.

6. A dying gust on a swing. A swing at rest is pushed by a gust whose force decays: $x'' + x = e^{-t}$, $x(0) = 0$, $x'(0) = 0$. Find $x(t)$ and the amplitude of the motion for large t.

Springs & Circuits

7. Car suspension. A car wheel assembly (mass 1) on a spring of stiffness 16 has a shock absorber with damping c: $x'' + cx' + 16x = 0$, $x(0) = 1$, $x'(0) = 0$ (scaled displacement). Find the critical damping and the motion at that value.

8. Washing-machine resonance. An unbalanced washing-machine drum drives its frame at exactly the natural frequency: $x'' + 25x = 2\cos 5t$, $x(0) = 0$, $x'(0) = 0$. Find $x(t)$ and describe the motion.

9. Charging an RLC circuit. A series RLC circuit with $L = 1$ H, $R = 6\ \Omega$, $C = 1/25$ F is connected to a 50 V battery: $q'' + 6q' + 25q = 50$, $q(0) = 0$, $q'(0) = 0$. Find $q(t)$, its steady state and its first peak.

Nonlinear Equations

10. Big swings are slower. A pendulum of length 1 m obeys $\theta'' = -9.8 \sin\theta$. Compare the period for release from rest at $\theta(0) = 0.3$ and at $\theta(0) = 2$.

11. Hanging cable. A flexible cable hanging under its own weight satisfies $y'' = 0.5\sqrt{(1 + (y')^2)}$ with lowest point $y(0) = 2$, $y'(0) = 0$. Find the shape $y(x)$ and the height at $x = 4$.

12. Escape velocity. In scaled units a projectile launched vertically from the Earth obeys $y'' = -1/(1 + y)^2$, $y(0) = 0$, $y'(0) = v$. Compare $v = 1.2$ and $v = 1.5$, and find the escape speed.

Boundary-Value Problems

13. Buckling column. A thin column of length 1 under axial load λ satisfies $y'' + \lambda y = 0$ with $y(0) = 0$ and $y(1) = 0$. Find the values of λ with nonzero solutions and the smallest (critical) load.

14. Sagging cable. A cable with uniform load between poles at $x = 0$ and $x = 4$ satisfies $y'' = 0.2$, $y(0) = 0$, $y(4) = 0$. Find $y(x)$, the starting slope and the maximum sag. (Shooting: try several initial slopes.)

15. Cooling fin. A cooling fin of length 1 satisfies $\theta'' = \theta$ (scaled), where θ is the temperature excess divided by its value at the base: $\theta(0) = 1$, and the tip is insulated: $\theta'(1) = 0$. Find $\theta(x)$, the needed slope $\theta'(0)$ and the tip value.

Linear Systems

16. Arms race. In a Richardson arms-race model, $x' = -x + 2y$, $y' = 2x - y$. Find the eigenvalues, classify the origin and describe what happens from $x(0) = 1$, $y(0) = 0.5$.

17. Two rooms cooling. Two connected rooms lose heat to the outdoors: $x' = -2x + y$, $y' = x - 2y$ (temperatures above outdoor). With $x(0) = 20$, $y(0) = 0$, find x, y and the largest temperature of room B.

18. LC circuit. An LC circuit with $L = 1$ H and $C = 1/4$ F gives the system $q' = i$, $i' = -4q$ with $q(0) = 1$, $i(0) = 0$. Describe the trajectory in the q–i plane, the period and the maximum current.

Solutions

Constant-Coefficient Equations

1. Door closer

- Characteristic equation $m^2 + 5m + 6 = 0 \to m = -2, -3$.
- $\theta = c_1 e^{-2t} + c_2 e^{-3t}$; $\theta(0) = 1.2$, $\theta'(0) = 0 \to c_1 = 3.6$, $c_2 = -2.4$.
- $\theta = 3.6e^{-2t} - 2.4e^{-3t} = e^{-3t}(3.6e^{t} - 2.4) > 0$ for all $t \geq 0$.
- Overdamped: the door creeps shut without passing $\theta = 0$.

> Answer: $m^2 + 5m + 6 = 0 \to m = -2, -3$ (two real roots) / $\theta = 3.6e^{-2t} - 2.4e^{-3t} > 0 \to$ no overshoot

2. Building sway

- $m^2 + 2m + 1 = 0 \to m = -1$ (repeated).
- $x = (c_1 + c_2 t)e^{-t}$; $x(0) = 0 \to c_1 = 0$; $x'(0) = 3 \to c_2 = 3$.
- $x = 3te^{-t}$; $x' = 3(1 - t)e^{-t} = 0$ at $t = 1$.
- Maximum displacement $3/e \approx 1.10$ (critically damped, no oscillation).

> Answer: $m = -1$ (double root) $\to x = 3te^{-t}$ / $x' = 0$ at $t = 1 \to$ max $x = 3/e \approx 1.10$

3. Potential between spheres

- Try $V = r^m$: $m(m - 1) + 2m = m(m + 1) = 0 \to m = 0, -1$.
- $V = c_1 + c_2/r$.
- $V(1) = 4$, $V(2) = 0 \to c_2 = 8$, $c_1 = -4$.
- $V = 8/r - 4$; e.g. $V(1.5) \approx 1.33$ V. (As an IVP: $V'(1) = -8$.)

> Answer: $V = r^m \to m(m - 1) + 2m = 0 \to m = 0, -1$ / $V = 8/r - 4$ ($V(1.5) \approx 1.33$ V)

Nonhomogeneous Equations

4. Bead sinking in oil

- Homogeneous: $m^2 + 2m = 0 \to y_c = c_1 + c_2 e^{-2t}$.
- Constant forcing with $m = 0$ a root $\to$ try $y_p = At$: $2A = 2 \to A = 1$.
- $y = t - (1 - e^{-2t})/2$.
- $y(4) \approx 3.50$; terminal speed $y' \to 1$.

> Answer: $y_p = t$ (undetermined coefficients) / $y = t - (1 - e^{-2t})/2$, $y(4) \approx 3.50$, speed $\to 1$

5. Marching on a bridge

- $y_c = c_1\cos 3t + c_2\sin 3t$; try $x_p = A\cos 2t$: $5A = 4 \to A = 4/5$.
- $x(0) = 0 \to c_1 = -4/5$; $x'(0) = 0 \to c_2 = 0$.
- $x = (4/5)(\cos 2t - \cos 3t)$.
- At $t = \pi$: $\cos 2\pi = 1$, $\cos 3\pi = -1 \to x = 1.6$, the largest possible.

> Answer: $x_p = (4/5)\cos 2t$ / $x = (4/5)(\cos 2t - \cos 3t)$, max $|x| = 1.6$ at $t = \pi$

6. A dying gust on a swing

- $y_c = c_1\cos t + c_2\sin t$; try $x_p = Ae^{-t}$: $2A = 1 \to A = 1/2$.
- $x(0) = 0 \to c_1 = -1/2$; $x'(0) = 0 \to c_2 = 1/2$.
- $x = (e^{-t} - \cos t + \sin t)/2$.
- For large t, $x \approx (\sin t - \cos t)/2 = (\sqrt{2}/2)\sin(t - \pi/4)$: amplitude ≈ 0.71.

> Answer: $x_p = e^{-t}/2$, $x = (e^{-t} - \cos t + \sin t)/2$ / $e^{-t} \to 0$, so amplitude $\sqrt{2}/2 \approx 0.71$ remains

Springs & Circuits

7. Car suspension

- $m^2 + cm + 16 = 0$ has a double root when $c^2 - 64 = 0 \to c = 8$.
- Then $m = -4$ (double): $x = (c_1 + c_2 t)e^{-4t}$.
- $x(0) = 1$, $x'(0) = 0 \to x = (1 + 4t)e^{-4t}$.
- $c < 8$: underdamped (bounces); $c > 8$: overdamped (slower return).

> Answer: Critical damping: $c^2 = 4 \cdot 16 \to c = 8$ / $x = (1 + 4t)e^{-4t}$; $c < 8$ bounces, $c > 8$ creeps

8. Washing-machine resonance

- $y_c = c_1\cos 5t + c_2\sin 5t$; $\cos 5t$ is already in $y_c \to x_p = t(A\cos 5t + B\sin 5t)$.
- Substituting: $10B = 2$, $A = 0 \to x_p = (t\sin 5t)/5$.
- Initial conditions give $c_1 = c_2 = 0$: $x = (t\sin 5t)/5$.
- Pure resonance: the amplitude $t/5$ grows without bound.

> Answer: Resonance: try $x_p = t(A\cos 5t + B\sin 5t)$ / $x = (t\sin 5t)/5 \to$ amplitude grows like $t/5$

9. Charging an RLC circuit

- $m^2 + 6m + 25 = 0 \to m = -3 \pm 4i$ (underdamped).
- $q_p = 50/25 = 2$.
- $q(0) = 0$, $q'(0) = 0 \to q = 2 - e^{-3t}(2\cos 4t + 1.5\sin 4t)$.
- $q \to 2$ C; first peak at $t = \pi/4$: $q \approx 2.19$ C.

> Answer: $m = -3 \pm 4i$, $q_p = 2$ / $q = 2 - e^{-3t}(2\cos 4t + 1.5\sin 4t)$, peak ≈ 2.19

Nonlinear Equations

10. Big swings are slower

- For small θ, $\sin\theta \approx \theta$: $\theta'' = -9.8\theta$, period $2\pi/\sqrt{9.8} \approx 2.01$ s.
- The nonlinear period is $T = 4K(\sin(\theta_0/2))/\sqrt{9.8}$.
- $\theta_0 = 0.3$: $T \approx 2.02$ s; $\theta_0 = 2$: $T \approx 2.67$ s.
- Unlike a linear spring, the pendulum period depends on amplitude.

Answer: Small swing: $\sin\theta \approx \theta \to T \approx 2\pi/\sqrt{9.8} \approx 2.01$ s / Big swing $\theta_0 = 2$: $T \approx 2.67$ s (period grows)

11. Hanging cable

- Missing y: let $u = y'$, so $u' = 0.5\sqrt{(1 + u^2)}$.
- Separate: $\operatorname{arcsinh} u = x/2 \to u = \sinh(x/2)$.
- Integrate: $y = 2\cosh(x/2) + C$; $y(0) = 2 \to C = 0$.
- Catenary $y = 2\cosh(x/2)$; $y(4) = 2\cosh 2 \approx 7.52$.

Answer: Set $u = y'$: $u' = 0.5\sqrt{(1 + u^2)} \to u = \sinh(x/2)$ / $y = 2\cosh(x/2)$ (catenary), $y(4) \approx 7.52$

12. Escape velocity

- Multiply by y' and integrate: $y'^2/2 - 1/(1 + y) = v^2/2 - 1$ (energy).
- The object escapes if y' never reaches 0, i.e. $v^2/2 - 1 \geq 0 \to v \geq \sqrt{2}$.
- $v = 1.2$: max height when $y' = 0 \to 1/(1 + y) = 0.28 \to y \approx 2.57$, then falls back.
- $v = 1.5 > \sqrt{2} \approx 1.41$: escapes.

Answer: Energy: $y'^2/2 - 1/(1 + y) = v^2/2 - 1$ / Escape iff $v \geq \sqrt{2} \approx 1.41 \to v = 1.5$ escapes

Boundary-Value Problems

13. Buckling column

- For $\lambda \leq 0$ only $y = 0$ works.
- For $\lambda = \alpha^2 > 0$: $y = c_2 \sin\alpha x$; $y(1) = 0$ needs $\sin\alpha = 0 \to \alpha = n\pi$.
- Eigenvalues $\lambda_n = n^2\pi^2$, eigenfunctions $\sin n\pi x$.
- Critical buckling load $\lambda_1 = \pi^2 \approx 9.87$.

Answer: Nonzero solution only for $\lambda = n^2\pi^2$ (eigenvalues) / Critical load $\lambda_1 = \pi^2 \approx 9.87$, shape $y = \sin\pi x$

14. Sagging cable

- Integrate twice: $y = 0.1x^2 + c_1x + c_2$.
- $y(0) = 0 \to c_2 = 0$; $y(4) = 0 \to 1.6 + 4c_1 = 0 \to c_1 = -0.4$.
- $y = 0.1x(x - 4)$, $y'(0) = -0.4$.
- Lowest point $x = 2$: sag 0.4.

Answer: $y = 0.1x^2 + c_1x$, $y(4) = 0 \to y'(0) = -0.4$ / $y = 0.1x(x - 4)$, max sag 0.4 at $x = 2$

15. Cooling fin

- $\theta = A\cosh(1 - x) + B\sinh(1 - x)$.
- $\theta'(1) = 0 \to B = 0$; $\theta(0) = 1 \to A = 1/\cosh 1$.
- $\theta = \cosh(1 - x)/\cosh 1$, $\theta'(0) = -\tanh 1 \approx -0.762$.
- Tip: $\theta(1) = 1/\cosh 1 \approx 0.648$.

> Answer: $\theta = \cosh(1 - x)/\cosh 1$, $\theta'(0) = -\tanh 1 \approx -0.762$ / Tip $\theta(1) = 1/\cosh 1 \approx 0.648$ (64.8% of the base)

Linear Systems

16. Arms race

- Matrix $[[-1, 2], [2, -1]]$: $\lambda = -1 \pm 2 \to 1, -3$.
- Eigenvectors $(1, 1)$ for $\lambda = 1$ and $(1, -1)$ for $\lambda = -3$.
- Opposite signs → saddle point (unstable).
- Any start with $x + y > 0$ grows along $(1, 1)$: the race escalates.

> Answer: Eigenvalues 1 and −3 → saddle point / Along (1, 1) it grows like e^t → escalation

17. Two rooms cooling

- $\lambda = -1$ (vector $(1, 1)$), $\lambda = -3$ (vector $(1, -1)$).
- $x = 10e^{-t} + 10e^{-3t}$, $y = 10e^{-t} - 10e^{-3t}$.
- $y' = 0 \to e^{2t} = 3 \to t = \ln 3/2 \approx 0.55$.
- $y_{max} \approx 3.85$; both rooms then cool toward outdoor temperature.

> Answer: Eigenvalues −1, −3 → stable node / $y = 10e^{-t} - 10e^{-3t}$, max ≈ 3.85 at $t \approx 0.55$

18. LC circuit

- Matrix $[[0, 1], [-4, 0]]$: $\lambda = \pm 2i \to$ center.
- $q = \cos 2t$, $i = -2\sin 2t$.
- Trajectory: $q^2 + i^2/4 = 1$ (ellipse), period $2\pi/2 = \pi$.
- Maximum current 2 A; energy moves between C and L forever.

> Answer: Eigenvalues $\pm 2i$ → center (closed ellipse) / $q^2 + i^2/4 = 1$: period π, max current 2