First-Order ODEs in Action

SnackLab Practice Set · First-order differential equations · 27 applied problems with solutions · snacklab.dev

Original practice problems written by SnackLab, organized by topic, from direction fields to systems. Every answer was checked numerically. Each problem is shown graphically in a SnackLab short; try it yourself in the SnackLab app (d/dx mode).

Direction Fields & Autonomous ODEs

1. Allee effect. A small animal population P(t) (in hundreds) follows $dP/dt = 0.2P(P - 1)(5 - P)$. Find all equilibrium solutions and classify each. Describe what happens if $P(0) = 0.8$ and if $P(0) = 1.2$.

2. Indoor temperature cycle. The indoor temperature T(t) of a building obeys Newton's law $T' = 0.5(A(t) - T)$, where the outdoor temperature is $A(t) = 5 + 3 \sin t$ (scaled units). Using the direction field, describe the long-run behavior for $T(0) = 0$ and $T(0) = 10$. What is the long-run amplitude of T?

3. Leaky tank. Water flows into a tank at a constant rate while leaking through a hole, so the depth h(t) satisfies $dh/dt = 1 - \sqrt{h}$. Find the equilibrium depth and decide whether it is stable. What happens if the tank starts empty, or at depth 4?

Separable Equations

4. Radioactive decay. A radioactive sample has mass 10 g. After 3 days, 8 g remain. Assuming $dm/dt = -km$, find the half-life and the time when only 1 g remains.

5. Coasting boat. A boat coasts after its engine is shut off at 5 m/s. Water drag gives $dv/dt = -0.1v^2$. When does the speed drop to 1 m/s, and how far has the boat traveled by then?

6. Seasonal population. A population with seasonal growth rate satisfies $dP/dt = 0.5P \cos t$, $P(0) = 2$. Find $P(t)$ and its largest and smallest values.

Linear Equations

7. Polluted lake. A lake receives a pollutant at the decaying rate $2e^{-0.5t}$ kg/day, and the pollutant is flushed out at the rate 0.5y kg/day: $y' = 2e^{-0.5t} - 0.5y$, $y(0) = 0$. Find $y(t)$ and the time of maximum pollution.

8. Savings with deposits. An account earns 5% interest compounded continuously and receives continuous deposits of \$2,000 per year. With A in units of \$10,000: $A' = 0.05A + 0.2$, $A(0) = 0$. When does the balance reach \$50,000?

9. RL circuit, AC source. A series RL circuit has $L = 1$ H, $R = 2\ \Omega$ and source $E(t) = 4 \sin t$ V, with $i(0) = 0$. Then $i' + 2i = 4 \sin t$. Find $i(t)$ and the amplitude of the steady-state current.

Exact Equations

10. Path of a particle. A particle moves in the plane along a curve satisfying $(2x + y)dx + (x + 2y)dy = 0$ and passes through $(1, 1)$. Find the curve and its highest point.

11. Hardening spring. A mass on a hardening spring satisfies $v\,dv/dx = -(x + x^3)$, where v is the velocity at position x. The mass is released from rest at $x = 1$. Find the energy curve in the (x, v)-plane and the maximum speed.

12. Streamline of a flow. Streamlines of a planar flow satisfy $(2y + 3x)dx + x\,dy = 0$. Find the streamline through $(1, 1)$ and the point where it crosses the x-axis.

Solutions by Substitution

13. Fish growth (Bertalanffy). The weight W(t) (kg) of a fish follows the von Bertalanffy model $dW/dt = 2W^{2/3} - W$, $W(0) = 1$. Find W(t), the limiting weight, and the time when $W = 4$.

14. Orthogonal trajectories. Field lines are perpendicular to the equipotential circles $x^2 + y^2 = 2cx$. Show that they satisfy $y' = 2xy/(x^2 - y^2)$, solve this equation, and find the field line through $(1, 1/2)$.

15. Swimming across a river. A swimmer starts at (1, 0) and always heads toward a dock at the origin with speed 2, while the river current carries her in the +y direction with speed 1. Her path satisfies $dy/dx = (y - \tfrac{1}{2}\sqrt{(x^2 + y^2)})/x$. Find the path. Does she reach the dock? What is her largest drift?

Euler's Method

16. Falling with air drag. A falling object satisfies $dv/dt = 9.8 - 0.2v^2$, $v(0) = 0$. Use Euler's method with $h = 0.1$ to estimate $v(0.5)$. Compare with the exact solution $v = 7\tanh(1.4t)$ and find the terminal speed.

17. Fishing at the limit. A fish population with constant harvesting satisfies $P' = 0.5P(1 - P/8) - 1$, $P(0) = 2$. Use Euler's method with $h = 0.5$ to estimate $P(2)$. Then solve exactly and find when the population dies out.

18. Seasonal inflow. The amount of pollutant $y(t)$ in a pond satisfies $y' = 1 + \sin t - 0.5y$, $y(0) = 0$. Use Euler's method with $h = 0.5$ to estimate $y(2)$, and describe the long-run behavior.

Linear Models

19. Carbon dating. Charcoal from an ancient fire pit contains 30% of the original amount of carbon-14. The half-life of C-14 is 5730 years. Estimate the age of the charcoal.

20. IV drug infusion. A drug is given by IV drip at 2 mg/h and is eliminated at a rate proportional to the amount present (rate constant 0.25 per hour). With $y(0) = 0$, find $y(t)$, the long-run amount, and when y reaches 90% of it.

21. Time of death. A body is found in a 20 °C room at 30 °C; one hour later it is 28 °C. Assuming Newton's law of cooling and a normal body temperature of 37 °C, estimate how long before discovery death occurred.

Nonlinear Models

22. Spread of a rumor. A rumor spreads in a town of 1000 people at a rate proportional to the product of those who have heard it and those who have not. Initially 10 people know it; after 2 days, 100 do. When will half the town know?

23. Tumor growth (Gompertz). The volume $y(t)$ of a tumor follows the Gompertz model $dy/dt = 0.5y(\ln 8 - \ln y)$, $y(0) = 1$. Find $y(t)$, its limiting size, and when it reaches half of that size.

24. Draining a cone. Water drains from an inverted conical tank through a hole at the vertex. By Torricelli's law the depth satisfies $dh/dt = -3h^{-3/2}$ (ft, min), with $h(0) = 4$. When is the tank empty?

Systems of ODEs

25. Two connected tanks. Two 10-L tanks: pure water enters A at 2 L/min, A→B at 3 L/min, B→A at 1 L/min, and B drains at 2 L/min. Tank A starts with 5 kg of salt, B with none. Set up the system and find the largest amount of salt in B.

26. Epidemic (SIR). In an SIR model (S, I in thousands) $S' = -0.5SI$, $I' = 0.5SI - I$, with $S(0) = 9.9$ and $I(0) = 0.1$. Find the peak number of infected people and the number who never get infected.

27. Lanchester combat. Two armies fight: $x' = -2y$, $y' = -x$ (thousands of soldiers), with $x(0) = 6$, $y(0) = 4$. Army Y is twice as effective per soldier. Who wins, and how many soldiers remain?

Solutions

Direction Fields & Autonomous ODEs

1. Allee effect

- Equilibria: $P = 0, 1, 5$ (zeros of the right side).
- Sign of P': negative on $(0, 1)$, positive on $(1, 5)$, negative on $(5, \infty)$.
- So $P = 0$ and $P = 5$ are stable (attractors), $P = 1$ is unstable (repeller).
- $P(0) = 0.8$ lies below the threshold $1 \to P$ decreases to 0 (extinction).
- $P(0) = 1.2$ lies above $1 \to P$ increases toward the carrying capacity 5.
- (This critical threshold is called the Allee effect.)

> Answer: P = 0 stable, P = 1 unstable, P = 5 stable; P(0) = 0.8 → dies out; P(0) = 1.2 → grows to 5

2. Indoor temperature cycle

- The equation is linear: $T' + 0.5T = 2.5 + 1.5 \sin t$.
- Every solution = transient $Ce^{-t/2}$ + one periodic steady state.
- Steady state: $T_p = 5 + (1.5/1.25)(0.5 \sin t - \cos t) = 5 + 1.342 \sin(t - 1.107)$.
- So from any start, T settles into oscillation between ≈ 3.66 and ≈ 6.34.
- Amplitude $3 \cdot 0.5/\sqrt{(0.25 + 1)} = 1.34 < 3$: the building damps and delays the outdoor swing.

> Answer: All solutions approach one periodic solution; T swings 5 ± 1.34 (smaller than outdoor ± 3), lagging

3. Leaky tank

- Equilibrium: $1 - \sqrt{h} = 0 \Rightarrow h = 1$.
- For $0 \le h < 1$, $h' > 0$ (level rises); for $h > 1$, $h' < 0$ (level falls).
- Hence $h = 1$ is asymptotically stable.
- $h(0) = 0$ rises toward 1; $h(0) = 4$ falls toward 1. Neither crosses $h = 1$.
- With inflow rate a, the equilibrium is $h = a^2$.

> Answer: Equilibrium h = 1 (inflow = outflow); Stable: from h(0) = 0 or 4 the level tends to 1

Separable Equations

4. Radioactive decay

- Separate: $dm/m = -k\,dt \Rightarrow m = 10e^{-kt}$.
- $m(3) = 8 \Rightarrow e^{-3k} = 0.8 \Rightarrow k = \ln(5/4)/3 \approx 0.07438$ per day.
- Half-life: $t_{1/2} = \ln 2 / k \approx 9.32$ days.
- $m = 1 \Rightarrow e^{-kt} = 0.1 \Rightarrow t = \ln 10 / k \approx 30.96$ days.

> Answer: $k = \ln(5/4)/3 \approx 0.0744$ /day, half-life ≈ 9.32 days; 1 g left at $t = \ln 10 / k \approx 31.0$ days

5. Coasting boat

- Separate: $dv/v^2 = -0.1\,dt \Rightarrow 1/v = 0.1t + 1/5 \Rightarrow v = 5/(1 + 0.5t)$.
- $v = 1 \Rightarrow 1 + 0.5t = 5 \Rightarrow t = 8$ s.
- Distance: $\int_0^8 5/(1 + 0.5t)\,dt = 10 \ln(1 + 0.5t)\,|_0^8 = 10 \ln 5 \approx 16.1$ m.
- $v > 0$ for all t, but the distance $10 \ln(1 + 0.5t)$ grows without bound.

> Answer: $v = 5/(1 + 0.5t) \to v = 1$ at $t = 8$ s; Distance $= 10 \ln 5 \approx 16.1$ m (never fully stops)

6. Seasonal population

- Separate: $dP/P = 0.5 \cos t\,dt \Rightarrow \ln P = 0.5 \sin t + C$.
- $P(0) = 2 \Rightarrow P = 2e^{0.5 \sin t}$.
- Max when $\sin t = 1$: $P = 2e^{0.5} \approx 3.30$; min when $\sin t = -1$: $P = 2e^{-0.5} \approx 1.21$.
- The population is periodic with period 2π (no long-term growth).

> Answer: $P = 2e^{0.5 \sin t}$; Oscillates between $2e^{-0.5} \approx 1.21$ and $2e^{0.5} \approx 3.30$

Linear Equations

7. Polluted lake

- Linear: $y' + 0.5y = 2e^{-0.5t}$, integrating factor $\mu = e^{0.5t}$.
- $(e^{0.5t}\,y)' = 2 \Rightarrow e^{0.5t}\,y = 2t \Rightarrow y = 2t\,e^{-0.5t}$.
- $y' = 2e^{-0.5t}(1 - 0.5t) = 0 \Rightarrow t = 2$.
- Maximum $y(2) = 4e^{-1} \approx 1.47$ kg.

> Answer: Integrating factor $e^{t/2}$: $y = 2t\,e^{-t/2}$; Peak at $t = 2$ days: $y = 4/e \approx 1.47$ kg

8. Savings with deposits

- Linear: $A' - 0.05A = 0.2$, $\mu = e^{-0.05t}$.
- $(e^{-0.05t}A)' = 0.2e^{-0.05t} \Rightarrow A = -4 + Ce^{0.05t}$; $A(0) = 0 \Rightarrow C = 4$.
- $A = 4(e^{0.05t} - 1)$.
- $A = 5 \Rightarrow e^{0.05t} = 2.25 \Rightarrow t = 20 \ln 2.25 \approx 16.2$ years.

> Answer: $A = 4(e^{0.05t} - 1)$ (× \$10,000); $A = 5$ at $t = 20 \ln 2.25 \approx 16.2$ years

9. RL circuit, AC source

- $\mu = e^{2t}$: $(e^{2t} i)' = 4e^{2t} \sin t$.
- $\int e^{2t} \sin t \, dt = e^{2t}(2 \sin t - \cos t)/5$.
- $i = 0.8(2 \sin t - \cos t) + Ce^{-2t}$; $i(0) = 0 \Rightarrow C = 0.8$.
- Transient $0.8e^{-2t} \to 0$; steady state $0.8(2 \sin t - \cos t)$ has amplitude $0.8\sqrt{5} = 4/\sqrt{5} \approx 1.79$ A.

> Answer: $i = 0.8(2 \sin t - \cos t) + 0.8e^{-2t}$; Transient dies out; amplitude $4/\sqrt{5} \approx 1.79$ A

Exact Equations

10. Path of a particle

- $M_y = 1 = N_x$, so the equation is exact.
- $F = x^2 + xy + y^2$: $F_x = 2x + y$, $F_y = x + 2y$.
- Through (1, 1): $x^2 + xy + y^2 = 3$, a rotated ellipse.
- Highest point: $dy/dx = 0 \Rightarrow 2x + y = 0 \Rightarrow y = -2x \Rightarrow 3x^2 = 3 \Rightarrow (-1, 2)$.

> Answer: Exact: $x^2 + xy + y^2 = 3$ (a tilted ellipse); Highest point $(-1, 2)$

11. Hardening spring

- Write $(x + x^3)dx + v \, dv = 0$: $M_v = 0 = N_x$, exact.
- $F = x^2/2 + x^4/4 + v^2/2 = C$ (total energy).
- At $x = 1$, $v = 0$: $C = 1/2 + 1/4 = 3/4$.
- Speed is largest at $x = 0$: $v^2/2 = 3/4 \Rightarrow v = \sqrt{1.5} \approx 1.22$.
- The closed curve shows periodic oscillation between $x = -1$ and $x = 1$.

> Answer: Exact: $v^2/2 + x^2/2 + x^4/4 = 3/4$ (energy); Max speed at $x = 0$: $v = \sqrt{1.5} \approx 1.22$

12. Streamline of a flow

- $M_y = 2$, $N_x = 1$: not exact. $(M_y - N_x)/N = 1/x$ depends only on $x \Rightarrow \mu = x$.
- $x(2y + 3x)dx + x^2 \, dy = 0$ is exact with $F = x^2y + x^3$.
- Through (1, 1): $x^2y + x^3 = 2 \Rightarrow y = 2/x^2 - x$.
- $y = 0 \Rightarrow x^3 = 2 \Rightarrow x = \sqrt[3]{2} \approx 1.26$.

> Answer: Not exact; $\mu = x$: $x^2y + x^3 = 2$; $y = 2/x^2 - x$ crosses $y = 0$ at $x = \sqrt[3]{2} \approx 1.26$

Solutions by Substitution

13. Fish growth (Bertalanffy)

- Bernoulli with $n = 2/3$. Let $u = W^{1/3}$: $u' = (1/3)W^{-2/3}W' = (2 - u)/3$.
- $u = 2 - Ce^{-t/3}$; $u(0) = 1 \Rightarrow C = 1$.
- $W = (2 - e^{-t/3})^3 \to 8$ kg as $t \to \infty$.
- $W = 4 \Rightarrow 2 - e^{-t/3} = \sqrt[3]{4} \Rightarrow t = -3\ln(2 - \sqrt[3]{4}) \approx 2.66$.

> Answer: Bernoulli, $u = W^{1/3}$: $W = (2 - e^{-t/3})^3$; $W \to 8$ kg; $W = 4$ at $t \approx 2.66$

14. Orthogonal trajectories

- Circles: $2x + 2yy' = 2c$ with $2c = (x^2 + y^2)/x \Rightarrow y' = (y^2 - x^2)/(2xy)$.
- Orthogonal slope: $y' = 2xy/(x^2 - y^2)$ (homogeneous).
- $y = ux$: $u + xu' = 2u/(1 - u^2) \Rightarrow (1 - u^2)du/(u(1 + u^2)) = dx/x$.
- Integrate: $\ln u - \ln(1 + u^2) = \ln x + C \Rightarrow x^2 + y^2 = 2ky$ (circles tangent to the x-axis).
- $(1, 1/2)$: $1 + 1/4 = k \Rightarrow x^2 + y^2 = 2.5y$, i.e. $x^2 + (y - 5/4)^2 = 25/16$.

> Answer: Homogeneous, $y = ux$: $x^2 + y^2 = 2ky$; Through $(1, 1/2)$: $x^2 + (y - 5/4)^2 = 25/16$

15. Swimming across a river

- Homogeneous: $y = ux$ gives $xu' = -\frac{1}{2}\sqrt{(1 + u^2)}$.
- $\int du/\sqrt{(1 + u^2)} = -\frac{1}{2}\ln x + C \Rightarrow \ln(u + \sqrt{(1 + u^2)}) = -\frac{1}{2}\ln x$ ($u = 0$ at $x = 1$).
- $u + \sqrt{(1 + u^2)} = x^{-1/2} \Rightarrow u = (x^{-1/2} - x^{1/2})/2$.
- $y = ux = \frac{1}{2}(\sqrt{x} - x^{3/2}) \to 0$ as $x \to 0$: she reaches the dock.
- $y' = 0 \Rightarrow x = 1/3 \Rightarrow$ max drift $y = \frac{1}{2}(1/\sqrt{3} - 1/(3\sqrt{3})) = 1/(3\sqrt{3}) \approx 0.192$.

> Answer: $y = \frac{1}{2}(\sqrt{x} - x^{3/2}) \to$ she reaches the dock; Max drift at $x = 1/3$: $y \approx 0.192$

Euler's Method

16. Falling with air drag

- $v_{n+1} = v_n + 0.1(9.8 - 0.2v_n^2)$.
- $v_1 = 0.98$, $v_2 = 1.9408$, $v_3 = 2.8455$, $v_4 = 3.6635$, $v_5 = 4.3751$.
- Exact: $v(0.5) = 7\tanh 0.7 \approx 4.2306$ (Euler overestimates, error ≈ 0.14).
- Terminal speed: $v' = 0 \Rightarrow v = \sqrt{(9.8/0.2)} = 7$ m/s.

> Answer: Euler: 0.98, 1.94, 2.85, 3.66, 4.38 → $v(0.5) \approx 4.38$; Exact $7\tanh(1.4t) = 4.23$; terminal speed 7 m/s

17. Fishing at the limit

- Euler: $P_1 = 1.875$, $P_2 = 1.7339$, $P_3 = 1.5734$, $P_4 = 1.3894 \approx P(2)$.
- Exactly: $0.5P - P^2/16 - 1 = -(P-4)^2/16$ ($P = 4$ is semi-stable).
- $u = P - 4$: $u' = -u^2/16 \Rightarrow 1/u = t/16 - 1/2 \Rightarrow P = 4 + 16/(t-8)$.
- $P(2) = 4 - 16/6 = 4/3 \approx 1.333$. $P = 0$ when $t - 8 = -4 \Rightarrow t = 4$.
- Lesson: below $P = 4$ even a small harvest drives the stock to extinction.

> Answer: Euler: 1.875, 1.734, 1.573, 1.389 (exact $P(2) = 4/3$); $P' = -(P-4)^2/16$: $P(0) = 2$ collapses at $t = 4$

18. Seasonal inflow

- Euler: $y_1 = 0.5$, $y_2 = 1.1147$, $y_3 = 1.7568$, $y_4 = 2.3163 \approx y(2)$.
- The equation is linear; its exact value is $y(2) \approx 2.2552$.
- Steady state: $y = 2 + 0.4 \sin t - 0.8 \cos t$, amplitude $1/\sqrt{1.25} \approx 0.894$.
- So y oscillates around 2 between ≈ 1.11 and ≈ 2.89.

> Answer: Euler: 0.5, 1.115, 1.757, 2.316 (exact $y(2) \approx 2.255$); Long run: $y \approx 2 + 0.894 \sin(t - 1.107)$

Linear Models

19. Carbon dating

- $A(t) = A_0 e^{-kt}$, $A(5730) = A_0/2 \Rightarrow k = \ln 2 / 5730 \approx 1.2097 \times 10^{-4}$.
- $A = 0.3A_0 \Rightarrow e^{-kt} = 0.3 \Rightarrow t = \ln(1/0.3)/k$.
- $t = 5730 \cdot \ln(10/3)/\ln 2 \approx 9{,}953$ years.
- (Graph: t in thousands of years, y = % remaining ÷ 10.)

> Answer: $k = \ln 2 / 5730 \approx 1.21 \times 10^{-4}$ /yr; $t = \ln(1/0.3)/k \approx 9{,}950$ years

20. IV drug infusion

- $y' + 0.25y = 2 \Rightarrow y = 8 + Ce^{-t/4}$; $y(0) = 0 \Rightarrow C = -8$.
- $y = 8(1 - e^{-t/4}) \to 8$ mg.
- $0.9 \cdot 8 \Rightarrow e^{-t/4} = 0.1 \Rightarrow t = 4 \ln 10 \approx 9.21$ h.
- Doubling the drip rate doubles the level but not the time to reach 90%.

> Answer: $y = 8(1 - e^{-t/4}) \to$ level 8 mg; 90% at $t = 4 \ln 10 \approx 9.2$ h

21. Time of death

- $T - 20 = 10e^{-kt}$ with $t = 0$ at discovery.
- $28 - 20 = 10e^{-k} \Rightarrow k = \ln(10/8) \approx 0.2231$ per hour.
- At death $T = 37$: $17 = 10e^{-kt} \Rightarrow t = -\ln 1.7 / k \approx -2.38$.
- Death ≈ 2.38 h ≈ 2 h 23 min before the body was found.
- (Graph: temperature in tens of °C, starting from the time of death.)

> Answer: $k = \ln(10/8) \approx 0.223$ /h; Death $\approx \ln 1.7 / k \approx 2.38$ h (2 h 23 min) before

Nonlinear Models

22. Spread of a rumor

- Logistic: $x = 1000/(1 + Ce^{-1000kt})$, $x(0) = 10 \Rightarrow C = 99$.
- $x(2) = 100 \Rightarrow 1 + 99e^{-2000k} = 10 \Rightarrow 1000k = \ln 11 / 2 \approx 1.199$.
- $x = 500 \Rightarrow 99e^{-1000kt} = 1 \Rightarrow t = \ln 99 / (1000k) = 2 \ln 99 / \ln 11 \approx 3.83$ days.
- (Graph: x in hundreds.)

> Answer: $x = 1000/(1 + 99e^{-1000kt})$, $1000k = (\ln 11)/2$; Half the town at $t = 2 \ln 99 / \ln 11 \approx 3.83$ days

23. Tumor growth (Gompertz)

- Let $z = \ln y$: $z' = y'/y = 0.5(\ln 8 - z)$, $z(0) = 0$.
- $z = \ln 8\,(1 - e^{-t/2}) \Rightarrow y = 8^{1 - e\hat{}\{-t/2}\} \to 8$.
- $y = 4 \Rightarrow 1 - e^{-t/2} = 2/3 \Rightarrow t = 2 \ln 3 \approx 2.20$.

> Answer: $\ln y = \ln 8\,(1 - e^{-t/2}) \to y \to 8$; $y = 4$ when $t = 2 \ln 3 \approx 2.20$

24. Draining a cone

- Separate: $h^{3/2}\,dh = -3\,dt \Rightarrow (2/5)h^{5/2} = -3t + C$.
- $h(0) = 4 \Rightarrow C = (2/5)\cdot 32 \Rightarrow h^{5/2} = 32 - 7.5t$.
- Empty: $h = 0 \Rightarrow t = 32/7.5 \approx 4.27$ min.
- The level falls faster near the bottom (narrow part of the cone).

> Answer: $h^{3/2}\,dh = -3\,dt \to h^{5/2} = 32 - 7.5t$; Empty at $t = 32/7.5 \approx 4.27$ min

Systems of ODEs

25. Two connected tanks

- $x' = -(3/10)x + (1/10)y$, $y' = (3/10)x - (3/10)y$.
- Eigenvalues: $(\lambda + 0.3)^2 = 0.03 \Rightarrow \lambda = -0.3 \pm 0.1732$: both negative $\Rightarrow (x, y) \to (0, 0)$.
- B is largest when $y' = 0 \Rightarrow y = x$ (on the phase curve).
- Numerically: $y_{max} \approx 1.96$ kg at $t \approx 3.8$ min.

> Answer: Eigenvalues $-0.3 \pm 0.1\sqrt{3}$, both negative → salt washes out; B peaks when $y = x$: ≈ 1.96 kg at $t \approx 3.8$ min

26. Epidemic (SIR)

- $dI/dS = (0.5SI - I)/(-0.5SI) = -1 + 2/S \Rightarrow I = -S + 2 \ln S + C$.
- $C = 10 - 2 \ln 9.9 \Rightarrow I = 10 - S + 2 \ln(S/9.9)$.
- I is largest at $S = 2$: $I_{max} = 8 + 2 \ln(2/9.9) \approx 4.80$ thousand.
- Epidemic ends when $I = 0$: $10 - S + 2 \ln(S/9.9) = 0 \Rightarrow S_\infty \approx 0.069$ thousand ≈ 69 people.

> Answer: $dI/dS = -1 + 2/S$: I_{max} at $S = 2 \approx 4.80$ (thousand); Never infected: $S_\infty \approx 0.069$ (≈ 69 people)

27. Lanchester combat

- $dx/dy = 2y/x \Rightarrow x\,dx = 2y\,dy \Rightarrow x^2 - 2y^2 = C$ (Lanchester square law).
- $C = 36 - 32 = 4 > 0 \Rightarrow$ Y reaches 0 first.
- When $y = 0$: $x^2 = 4 \Rightarrow x = 2$ thousand survivors.
- Numerically the battle ends at $t \approx 1.25$. (Y would need effectiveness above 2.25 to win.)

> Answer: $dy/dx = x/(2y) \to x^2 - 2y^2 = 4$ (square law); X wins with 2 thousand left ($t \approx 1.25$)