Laplace Transform in Action

SnackLab Practice Set · Laplace transform · 12 applied problems with solutions · snacklab.dev

Original practice problems written by SnackLab, organized by topic: transforms and inverse transforms, shifting theorems, convolution and periodic functions, and the Dirac delta function. Every answer was checked symbolically. Each problem is shown graphically in a SnackLab short; try it yourself in the SnackLab app (L mode), where the upper pane shows f(t) and the lower pane shows F(s).

Transforms & Inverse Transforms

1. Charging a capacitor. The voltage across a charging capacitor has Laplace transform $V(s) = 10/(s(s + 2))$. Find v(t), the final voltage, and the time at which $v = 4$ V.

2. Drug in the bloodstream. The concentration of a drug in the blood has Laplace transform $C(s) = (3s + 7)/((s + 1)(s + 3))$. Find C(t) and its value at $t = 1$.

3. Shock absorber. After a car hits a bump, the displacement of its body has Laplace transform $X(s) = (s + 4)/(s^2 + 4s + 13)$. Find x(t) and the first time $t > 0$ at which $x = 0$.

Shifting Theorems

4. Heater switched on late. A heater is switched on at $t = 2$ and the temperature rise of a room has Laplace transform $T(s) = 3e^{-2s}/(s(s + 1))$. Find $T(t)$ and the temperature rise at $t = 4$.

5. Pushing for two seconds. A constant force of 4 N acts on a cart from $t = 1$ s to $t = 3$ s. Write the force with unit step functions, find its Laplace transform $F(s)$, and find the total impulse.

6. Fading vibration. The vibration of a struck metal bar is modeled by $f(t) = e^{-t} \sin 2t$. Find $F(s)$ and use it to compute the area under $f(t)$ for $t \geq 0$.

Convolution & Periodic Functions

7. Resonance builds up. A spring driven at its natural frequency has displacement with Laplace transform $X(s) = 4s/(s^2 + 4)^2$. Find $x(t)$ and the size of the oscillation around $t = 5$.

8. Tank with a varying inflow. A tank starts empty. Water flows in at rate $1 + \sin t$ and drains at a rate equal to the amount y present: $y' + y = 1 + \sin t$, $y(0) = 0$. Write y as a convolution, find $y(t)$ and describe the long-term behavior.

9. Square-wave voltage. A switch applies 3 V for one second and 0 V for the next second, repeating forever (period 2). Find the Laplace transform $E(s)$ and the average voltage.

Dirac Delta Function

10. Hammer hits a spring. A mass on a spring, at rest, is struck by a hammer at $t = 1$: $y'' + 9y = 2\delta(t - 1)$, $y(0) = y'(0) = 0$. Find $y(t)$ and the amplitude of the motion.

11. A second injection. The amount of a drug in the body starts at 2 and decays at a rate equal to the amount. At $t = 2$ a dose of 4 is injected instantly: $y' + y = 4\delta(t - 2)$, $y(0) = 2$. Find $y(t)$ and the amount just after the injection.

12. Kicking a damped spring. A damped spring at rest receives a sharp kick at $t = 1$: $y'' + 2y' + 5y = \delta(t - 1)$, $y(0) = y'(0) = 0$. Find $y(t)$ and the largest displacement.

Solutions

Transforms & Inverse Transforms

1. Charging a capacitor

- Partial fractions: $10/(s(s + 2)) = 5/s - 5/(s + 2)$.
- $v(t) = 5 - 5e^{-2t} = 5(1 - e^{-2t})$; as $t \to \infty$, $v \to 5$ V.
- $5(1 - e^{-2t}) = 4 \to e^{-2t} = 1/5 \to t = (\ln 5)/2 \approx 0.80$.
- In general $10/(s(s + a))$ gives final value $10/a$ (the slider in the video).

> Answer: $10/(s(s + 2)) = 5/s - 5/(s + 2) \to v = 5(1 - e^{-2t})$ / $v = 4$ at $t = (\ln 5)/2 \approx 0.80$, final value 5 V

2. Drug in the bloodstream

- Write $(3s + 7)/((s + 1)(s + 3)) = A/(s + 1) + B/(s + 3)$.
- Cover-up: $A = (-3 + 7)/(-1 + 3) = 2$, $B = (-9 + 7)/(-3 + 1) = 1$.
- $C(t) = 2e^{-t} + e^{-3t}$.
- $C(1) = 2/e + e^{-3} \approx 0.786$.

> Answer: $A/(s + 1) + B/(s + 3)$: $A = 2$, $B = 1$ / $C = 2e^{-t} + e^{-3t}$, $C(1) \approx 0.79$

3. Shock absorber

- $s^2 + 4s + 13 = (s + 2)^2 + 9$, and $s + 4 = (s + 2) + 2$.
- $X(s) = (s + 2)/((s + 2)^2 + 9) + (2/3)\cdot 3/((s + 2)^2 + 9)$.
- First shifting theorem: $x(t) = e^{-2t}(\cos 3t + (2/3) \sin 3t)$.
- $x = 0$ when $\tan 3t = -3/2 \to 3t = \pi - \arctan 1.5 \to t \approx 0.72$.

> Answer: $(s + 2)/((s + 2)^2 + 9) + (2/3)\cdot 3/((s + 2)^2 + 9)$ / $x = e^{-2t}(\cos 3t + (2/3) \sin 3t)$, first zero $t \approx 0.72$

Shifting Theorems

4. Heater switched on late

- Without the delay: $3/(s(s + 1)) = 3/s - 3/(s + 1) \to 3(1 - e^{-t})$.
- Second shifting theorem: $e^{-2s}F(s) \leftrightarrow u(t - 2)f(t - 2)$.
- $T(t) = 3u(t - 2)(1 - e^{-(t-2)})$; $T = 0$ until $t = 2$.
- $T(4) = 3(1 - e^{-2}) \approx 2.59$.

> Answer: $e^{-2s} \to$ shift by 2: $T = 3u(t - 2)(1 - e^{-(t-2)})$ / $T(4) = 3(1 - e^{-2}) \approx 2.59$

5. Pushing for two seconds

- $f(t) = 4[u(t - 1) - u(t - 3)]$.
- $L\{u(t - a)\} = e^{-as}/s \to F(s) = 4(e^{-s} - e^{-3s})/s$.
- Impulse $= \int f\,dt$ over $t \geq 0 = F(0)$.
- $\lim (s \to 0)\ 4(e^{-s} - e^{-3s})/s = 4 \cdot (3 - 1) = 8$ N·s.

> Answer: $F(s) = 4(e^{-s} - e^{-3s})/s$ / Impulse = area = $F(0) = \lim (s \to 0)\ F(s) = 8$

6. Fading vibration

- $L\{\sin 2t\} = 2/(s^2 + 4)$.
- First shifting theorem: $L\{e^{-t}g(t)\} = G(s + 1) \to F(s) = 2/((s + 1)^2 + 4)$.
- Area $= \int e^{-t} \sin 2t\,dt$ over $t \geq 0 = F(0) = 2/5$.

> Answer: s-shift: $2/(s^2 + 4) \to 2/((s + 1)^2 + 4)$ / Area = $\int f\,dt$ over $t \geq 0 = F(0) = 2/5$

Convolution & Periodic Functions

7. Resonance builds up

- Multiplying by t: $L\{t\,f(t)\} = -F'(s)$.
- With $f = \sin 2t$: $-d/ds[2/(s^2 + 4)] = 4s/(s^2 + 4)^2$.
- So $x(t) = t \sin 2t$: the envelope $\pm t$ grows without bound (resonance).
- Near $t = 5$ the amplitude is about 5.

> Answer: $L\{t \sin 2t\} = -d/ds[2/(s^2 + 4)] = 4s/(s^2 + 4)^2$ / $x = t \sin 2t \to$ amplitude 5 at $t = 5$

8. Tank with a varying inflow

- Transform: $(s + 1)Y = 1/s + 1/(s^2 + 1) \to Y = (1/(s + 1)) \cdot (1/s + 1/(s^2 + 1))$.
- Convolution theorem: $y = e^{-t} * (1 + \sin t) = \int e^{-(t-\tau)}(1 + \sin \tau)\,d\tau$ from 0 to t.
- $y = 1 + (\sin t - \cos t)/2 - e^{-t}/2$.
- For large t, y oscillates between $1 - \sqrt{2}/2 \approx 0.29$ and $1 + \sqrt{2}/2 \approx 1.71$.

> Answer: $Y = (1/(s + 1)) \cdot (1/s + 1/(s^2 + 1))$ / $y = 1 + (\sin t - \cos t)/2 - e^{-t}/2 \to$ swings 1 ± 0.71

9. Square-wave voltage

- Periodic function: $E(s) = (1/(1 - e^{-2s})) \int e^{-st}E(t)\,dt$ over one period.
- One period: $\int 3e^{-st}\,dt$ from 0 to $1 = 3(1 - e^{-s})/s$.
- $E(s) = 3(1 - e^{-s})/(s(1 - e^{-2s})) = 3/(s(1 + e^{-s}))$.
- Average $= \lim (s \to 0)\ sE(s) = 3/2$ V.

> Answer: $E(s) = 3(1 - e^{-s})/(s(1 - e^{-2s})) = 3/(s(1 + e^{-s}))$ / Average = $\lim (s \to 0)\ sE(s) = 3/2$

Dirac Delta Function

10. Hammer hits a spring

- $L\{\delta(t-1)\} = e^{-s}$: $(s^2+9)Y = 2e^{-s}$.
- $Y = 2e^{-s}/(s^2+9) = (2/3)e^{-s}\cdot 3/(s^2+9)$.
- $y = (2/3)u(t-1)\sin 3(t-1)$: still until $t = 1$, then oscillates.
- Amplitude $2/3$.

> Answer: $Y = 2e^{-s}/(s^2+9)$ / $y = (2/3)u(t-1)\sin 3(t-1)$, amplitude $2/3$

11. A second injection

- $sY - 2 + Y = 4e^{-2s} \to Y = 2/(s+1) + 4e^{-2s}/(s+1)$.
- $y = 2e^{-t} + 4u(t-2)e^{-(t-2)}$.
- Just before $t = 2$: $2e^{-2} \approx 0.27$; the impulse adds a jump of 4.
- Just after: $4 + 2e^{-2} \approx 4.27$.

> Answer: $Y = 2/(s+1) + 4e^{-2s}/(s+1)$ / $y = 2e^{-t} + 4u(t-2)e^{-(t-2)}$, just after: ≈ 4.27

12. Kicking a damped spring

- $(s^2+2s+5)Y = e^{-s} \to Y = e^{-s}/((s+1)^2+4)$.
- $y = \tfrac{1}{2}u(t-1)e^{-(t-1)}\sin 2(t-1)$.
- With $\tau = t-1$: $d/d\tau[e^{-\tau}\sin 2\tau] = 0 \to \tan 2\tau = 2 \to \tau \approx 0.554$.
- Largest displacement $y \approx 0.257$ at $t \approx 1.55$.

> Answer: $y = \tfrac{1}{2}u(t-1)e^{-(t-1)}\sin 2(t-1)$ / peak at $t = 1 + (\arctan 2)/2 \approx 1.55$, $y \approx 0.257$